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## COVARIANT EXPANSION OF A MODULAR FORM.

BY OLIVER E. GLENN.

A complete system of covariants of the total group  $G$  of linear transformations with integral coefficients modulo  $p$ , a prime number, is composed of\*

$$L = x_1^p x_2 - x_1 x_2^p,$$

$$Q = (x_1^{p^2} x_2 - x_1 x_2^{p^2}) \div L = x_1^{p(p-1)} + x_1^{(p-1)(p-1)} x_2^{p-1} + \dots + x_2^{p(p-1)}.$$

Consider a binary form, of order  $m$ , whose coefficients are independent variables:

$$f_m = (a_0, a_1, \dots, a_m)(x_1, x_2)^m = a_0 x_1^m + a_1 x_1^{m-1} x_2 + \dots.$$

We propose to treat the problem of determining modular covariants  $\phi_1, \phi_2$  of  $f_m$  such that the following congruence will hold identically in the  $a$ 's and in the  $x$ 's:

$$(1) \quad f_m \equiv Q\phi_1 + L\phi_2 \pmod{p}.$$

Regarding the forms  $\phi_1, \phi_2$  in a relation like (1) to be general forms with undetermined coefficients, of respective orders  $m - p^2 + p, m - p - 1$ , it is evident that when  $m = p^2$  the identity (1) implies  $m + 1$  linear non-homogeneous equations between these coefficients. These linear equations are consistent, and hence  $\phi_1, \phi_2$  are uniquely determined. In fact, in order to prove the consistency of this system of equations we have only to construct their matrix  $M$ , the elements of  $M$  being all 0, 1 or  $-1$ . These elements are arranged in  $M$  by a simple law such that it is immediately evident that elementary transformations will reduce all elements to zero, excepting those in the principal diagonal, and that all in the diagonal will be  $\equiv 1 \pmod{p}$ . Hence if  $D$  is the determinant of  $M$ ,  $D \not\equiv 0 \pmod{p}$ .

When, as in the case  $m = p^2$  just mentioned, the quantics  $\varphi_1, \varphi_2$  are uniquely determined, they are readily proved† to be formal covariants modulo  $p$  of  $f_m$ . Also, for a series of particular orders  $m$ , such that  $m > p^2$ , I have determined, non-uniquely, covariant pairs  $\varphi_1, \varphi_2$  satis-

\* Dickson, Transactions Amer. Math. Society, vol. 12 (1911), p. 75; and Madison Colloquium Lectures, 1913, p. 33.

† Cf. O. E. Glenn, Transactions Amer. Math. Society, vol. 18 (1917), p. 460.

fying (1). We shall give in tables computed covariants  $\varphi_1, \varphi_2$  leading to an identity (1), for various cases in which  $m \geq p^2$ .

Whenever we have covariants\*  $\varphi_1$  and  $\varphi_2$  of  $f_m$  for which (1) is an identity, we shall call (1) a covariant expansion, and  $\varphi_1$  a principal covariant of  $f_m$ ; its seminvariant leader being  $a_0$ . In general the covariant  $\varphi_1$  leading to an expansion (1) is not unique. But any two principal covariants are linearly dependent modulo  $L$ ; indeed they all possess their real roots in common.† For, if  $\varphi_1, \varphi_1'$  are two such covariants their difference  $\varphi_1' - \varphi_1$ , being a covariant divisible by  $x_2$ , contains the factor  $L$ , since the real points on the modular line are conjugate under  $G$ . Thus in the expansion corresponding to  $\varphi_1'$ , viz.,

$$(2) \quad f_m \equiv Q\varphi_1' + L\varphi_2' \pmod{p},$$

we may substitute

$$(3) \quad \varphi_1' \equiv \varphi_1 + L\psi \pmod{p},$$

where  $\psi$  is a formal covariant mod  $p$  of order  $m - p^2 - 1$ ; and so (2) becomes

$$f_m \equiv Q\varphi_1 + L(Q\psi + \varphi_2') \pmod{p},$$

and the principal covariant in this expansion is  $\varphi_1$ . The latter expansion is identical with (1); for,  $\varphi_2$  may always be determined by dividing  $f_m - Q\varphi_1$  by  $L$ , and the quotient modulo  $p$  is unique. Hence the covariant expansions are all transformable into a fixed one by congruences of type (3).

In case  $m$  is sufficiently large,  $\varphi_1$  and  $\varphi_2$  will be of order  $\geq p^2$ , and then, under restrictions similar to those described above for covariant expansions of  $f_m$ ,  $\varphi_1, \varphi_2$  may themselves be developed in covariant expansions; also, if only one of the covariants  $\varphi_1, \varphi_2$  is of order  $\geq p^2$ , this one may be developed according to any expansion (1) known to exist for its order. Thus we arrive at a covariant expansion of  $f_m$  more explicit than (1), viz.,

$$(4) \quad f_m \equiv Q^a\varphi_1 + Q^{\beta-1}L\chi + \cdots + QL^{\gamma-1}\psi + L^\delta\omega \pmod{p},$$

in which the coefficient forms  $\varphi_1, \chi, \cdots, \psi, \omega$  are covariants of  $f_m$  of orders  $< p^2$ ; the principal covariant being  $\varphi_1$  led by  $a_0$ . This expansion is not unique.

The existence and explicit form of the principal formal covariants modulo 2 for the general order  $m$  were demonstrated by the present writer

\* The covariancy of  $\varphi_1$  evidently implies that of  $\varphi_2$ .

† Since the roots of  $L \equiv 0 \pmod{p}$  are the real residues mod  $p$ , and those of  $Q \equiv 0$  are the Galoisian imaginaries which are roots of irreducible quadratic congruences mod  $p$ , the real roots of  $f_m \equiv 0 \pmod{p}$  are also roots of  $\varphi_1 \equiv 0$ , and such imaginary roots of  $f_m \equiv 0$  are also roots of  $\varphi_2 \equiv 0 \pmod{p}$  (cf. Dickson, Madison Colloquium Lectures, p. 37).

in the Transactions of the American Mathematical Society, volume 17 (p. 545). For  $m$  even,  $\varphi_1$  is the covariant

$$K_2 = a_0x_1^2 + (a_1 + \cdots + a_{m-1})x_1x_2 + a_mx_2^2,$$

while for  $m$  odd it was proved that  $\varphi_1$  is a cubic covariant

$$K_{m3} = a_0x_1^3 + I_1x_1^2x_2 + I_2x_1x_2^2 + a_mx_2^3,$$

where  $I_1, I_2$  are of somewhat complex general structure but such that  $I_1 + I_2$  is congruent to the invariant  $a_1 + \cdots + a_{m-1}$ . Illustrations of  $K_2$  and  $K_{m3}$  are contained in the lists below, showing expansions modulo 2 for forms of the first eleven orders, and of the forms of orders 9, 10, 11 when the modulus is 3. The existence of the principal covariants in the latter three cases was established by methods similar to those employed for the modulus 2, but for  $m = 11$ , in order to exhibit a complete set of principal covariants in one formula, by retaining a parameter  $\lambda$  in their coefficients,  $\varphi_1$  was assumed in its general form and the conditions for its invariancy under the induced group were imposed. This method required the solution of a set of linear congruences in twenty unknowns, and was not brief, but, once these covariants are found, the direct verification of their covariancy is easy.

#### Tables.

We employ the notation  $(hijk \cdots)$  for the sum

$$a_h + a_i + a_j + a_k + \cdots$$

$$p = 2, \quad m = 1, 2, 3.$$

The forms  $f_1, f_2, f_3$  are irreducible.

$$p = 2, \quad m = 2^2.$$

$$f_4 \equiv QK_2 + L(K_1 + C_{101}) \pmod{2}.$$

$$K_2 \equiv (0)x_1^2 + (123)x_1x_2 + (4)x_2^2,$$

$$K_1 \equiv (0123)x_1 + (1234)x_2,$$

$$C_{101} \equiv (1)x_1 + (3)x_2.$$

$$p = 2, \quad m = 5.$$

$$f_5 \equiv QK_{53} + L(K_2 + C_{102}) \pmod{2}.$$

$$K_{53} \equiv (0)x_1^3 + (12)x_1^2x_2 + (34)x_1x_2^2 + (5)x_2^3,$$

$$K_2 \equiv (0)x_1^2 + (1234)x_1x_2 + (5)x_2^2,$$

$$C_{102} \equiv (2)x_1^2 + (3)x_2^2.$$

$$p = 2, \quad m = 6.$$

$$f_6 \equiv Q^2K_2 + LK_3 \pmod{2}.$$

$$K_2 \equiv (0)x_1^2 + (12345)x_1x_2 + (6)x_2^2,$$

$$K_3 \equiv (2345)x_1^3 + (03456)x_1^2x_2 + (01236)x_1x_2^2 + (1234)x_2^3.$$

$$p = 2, \quad m = 7.$$

$$f_7 \equiv Q^2K_{73} + LQM + L^2J \pmod{2}.$$

$$K_{73} \equiv (0)x_1^3 + (124)x_1^2x_2 + (356)x_1x_2^2 + (7)x_2^3,$$

$$M \equiv (24)x_1^2 + (35)x_2^2,$$

$$J \equiv (02356)x_1 + (12457)x_2.$$

$$p = 2, \quad m = 8.$$

$$f_8 \equiv Q^3K_2 + QLK_{53}' + L^2(K_2' + C_{102}') \pmod{2}.$$

$$K_2 \equiv (0)x_1^2 + (1234567)x_1x_2 + (8)x_2^2,$$

$$K_{53}' \equiv (0234567)x_1^3 + (038)x_1^2x_2 + (058)x_1x_2^2 + (1234568)x_2^3,$$

$$K_2' \equiv (0234567)x_1^2 + (35)x_1x_2 + (1234568)x_2^2,$$

$$C_{102}' \equiv (123)x_1^2 + (567)x_2^2.$$

$$p = 2, \quad m = 9.$$

$$f_9 \equiv Q^3K_{93} + Q^2LK_2' + L^2K_3' \pmod{2}.$$

$$K_{93} \equiv (0)x_1^3 + (1234)x_1^2x_2 + (5678)x_1x_2^2 + (9)x_2^3,$$

$$K_2' \equiv (0234)x_1^2 + (124578)x_1x_2 + (5679)x_2^2,$$

$$K_3' \equiv (046)x_1^3 + (157)x_1^2x_2 + (248)x_1x_2^2 + (359)x_2^3.$$

$$p = 2, \quad m = 10.$$

$$f_{10} \equiv Q^4K_2 + Q^2LK_{73}' + QL^2M' + L^3J' \pmod{2}.$$

$$K_2 \equiv (0)x_1^2 + (123456789)x_1x_2 + (10)x_2^2,$$

$$K_{73}' \equiv (23456789)x_1^3 + (0124510)x_1^2x_2 + (0568910)x_1x_2^2 + (12345678)x_2^3,$$

$$M' \equiv (01236789)x_1^2 + (123478910)x_2^2,$$

$$J' \equiv (0235689)x_1 + (12457810)x_2.$$

$$p = 2, \quad m = 11.$$

$$f_{11} \equiv Q^4 K_{113} + Q^3 L K_2' + Q L^2 K_{53}'' + L^3 (K_2'' + C_{102}'') \pmod{2}.$$

$$K_{113} \equiv (0)x_1^3 + (12458)x_1^2x_2 + (367910)x_1x_2^2 + (11)x_2^3,$$

$$K_2' \equiv (2458)x_1^2 + (56)x_1x_2 + (3679)x_2^2,$$

$$K_{53}'' \equiv (2357910)x_1^3 + (02341011)x_1^2x_2 + (0178911)x_1x_2^2 + (124689)x_2^3,$$

$$K_2'' \equiv (2357910)x_1^2 + (123478910)x_1x_2 + (124689)x_2^2,$$

$$C_{102}'' \equiv (035678910)x_1^2 + (123456811)x_2^2.$$

$$p = 3, \quad m = 3^2.$$

$$f_9 \equiv Q\varphi_1 + L\varphi_2 \pmod{3}.$$

$$\varphi_1 \equiv (0)x_1^3 + (1357)x_1^2x_2 + (2468)x_1x_2^2 + (9)x_2^3,$$

$$\varphi_2 \equiv 2(a_3 + a_5 + a_7)x_1^5 + 2(a_0 + a_4 + a_6 + a_8)x_1^4x_2$$

$$+ (2a_1 + 2a_3 + a_5 + a_7 + 2a_9)x_1^3x_2^2$$

$$+ (a_0 + 2a_2 + 2a_4 + a_6 + a_8)x_1^2x_2^3$$

$$+ (a_1 + a_3 + a_5 + a_9)x_1x_2^4 + (a_2 + a_4 + a_6)x_2^5.$$

$$p = 3, \quad m = 10.$$

$$f_{10} \equiv Q\varphi_1 + L\varphi_2 \pmod{3}.$$

$$\varphi_1 \equiv a_0x_1^4 + (a_1 + a_3 + 2a_5)x_1^3x_2 + (a_2 + a_4 + a_6 + a_8)x_1^2x_2^2$$

$$+ (2a_5 + a_7 + a_9)x_1x_2^3 + a_{10}x_2^4,$$

$$\varphi_2 \equiv (2a_3 + a_5)x_1^6 + 2(a_0 + a_4 + a_6 + a_8)x_1^5x_2$$

$$+ 2(a_1 + a_3 + a_7 + a_9)x_1^4x_2^2$$

$$+ (a_0 + 2a_2 + 2a_4 + a_6 + a_8 + 2a_{10})x_1^3x_2^3$$

$$+ (a_1 + a_3 + a_7 + a_9)x_1^2x_2^4 + (a_2 + a_4 + a_6 + a_{10})x_1x_2^5$$

$$+ (2a_5 + a_7)x_2^6.$$

$$p = 3, \quad m = 11.$$

The abbreviations employed are as follows:  $\lambda$  is any least residue modulo 3;  $S$  represents the seminvariant  $a_1 + a_3 + a_5 + a_7 + a_9$ , and  $T$  the anti-seminvariant  $a_2 + a_4 + a_6 + a_8 + a_{10}$ .

$$f_{11} \equiv Q\varphi_1 + L\varphi_2 \pmod{3}.$$

$$\begin{aligned}
\varphi_1 &\equiv a_0 x_1^5 + (a_1 + a_3 + \lambda S) x_1^4 x_2 + (a_2 + a_4 + a_6 - \lambda T) x_1^3 x_2^2 \\
&\quad + (a_5 + a_7 + a_9 - \lambda S) x_1^2 x_2^3 + (a_8 + a_{10} + \lambda T) x_1 x_2^4 + a_{11} x_2^5, \\
\varphi_2 &\equiv (-a_3 - \lambda S) x_1^7 + (-a_0 - a_4 - a_6 + \lambda T) x_1^6 x_2 - (\lambda + 1) S x_1^5 x_2^2 \\
&\quad + (a_0 - a_6 + (\lambda + 2) T) x_1^4 x_2^3 + (a_5 - a_{11} - (\lambda + 2) S) x_1^3 x_2^4 \\
&\quad + (\lambda + 1) T x_1^2 x_2^5 + (a_5 + a_7 + a_{11} - \lambda S) x_1 x_2^6 + (a_8 + \lambda T) x_2^7.
\end{aligned}$$

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